

RIGIDITY OF AMN VECTOR SPACES

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Abstract: *A metric vector space is asymptotically metrically normable (AMN) if there exists a norm asymptotically isometric to the distance. We prove that AMN vector spaces are rigid in the class of metric vector spaces under asymptotically isometric perturbations. This result follows from a general metric normability criterium. If the distance is translation invariant and satisfies an approximate multiplicative condition then there exists a lipschitz equivalent norm. Furthermore, we give necessary and sufficient conditions for the distance to be asymptotically isometric to the norm.*

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Introduction.

The geometrization of algebraic structures is a fruitful modern idea born from the collusion of synthetic geometry and classical algebra. A successful example is Gromov's notion of hyperbolic groups that has shed new light on classical group theory (see [Gr], [GH]). In this article we investigate metric vector spaces from a metrical point of view.

Let (E, d) be a metric space. The distance d is *asymptotically isometric* to a distance δ if for any $C_1 > 1$ there exists $C_2 \geq 0$ such that for any $x, y \in E$ we have

$$C_1^{-1}\delta(x, y) - C_2 \leq d(x, y) \leq C_1\delta(x, y) + C_2.$$

Two metric spaces (E_1, d_1) and (E_2, d_2) are asymptotically isometric if there exists a one-to-one correspondence $\varphi : E_1 \rightarrow E_2$ such that d_1 is asymptotically isometric to the distance $\delta = \varphi^* d_2$,

$$\delta(x, y) = d_2(\varphi(x), \varphi(y)) .$$

A metric vector space (E, d) is a topological vector space whose topology is generated by the distance d .

Definition 1. *A metric space (E, d) is asymptotically metrically normable (AMN) if there exists a norm $\| \cdot \|$ on E such that $(E, \| \cdot \|)$ is asymptotically isometric to (E, d) .*

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Our main theorem is:

Theorem 1. *AMN vector spaces are asymptotically isometrically rigid. More precisely, let (E, d) be a metric vector space. If E is asymptotically isometric to an AMN vector space, then E is AMN.*

Theorem 1 is a consequence of a metric normability criterium for metric vector spaces (theorems 2 and 3.)

It is well known that a Hausdorff topological vector space is metrizable if and only if the origin has a countable neighborhood base. In this case there exists a translation invariant distance generating the topology (see for example [Sch] p.28.) A Hausdorff topological vector space is normable if and only if the origin has a bounded convex neighborhood ([Sch] p.41.) Recall in a topological vector space a set A is bounded if for each neighborhood U of 0 there exists a scalar λ such that $A \subset \lambda U$.

These conditions are sharp but not always useful. For instance, given a distance d defining the topological vector space structure there is no effective way of determining the existence of a convex neighborhood of 0. Also an asymptotically isometric perturbation does not preserve convex sets (even those at "infinity".) The purpose of the following theorems is to exhibit explicit metric conditions on the distance that imply the existence of a lipschitz equivalent norm. Similar ideas were used by the author in the study of Hölder absolute values over a field (see [Mu].)

From now on we consider vector spaces over a locally compact valued field K of characteristic zero ($\mathbf{Q} \subset K$) and such that $(\mathbf{Q}, |\cdot|)$ is archimedian. From the classification of locally compact fields (see for example [We] chapter I.3) we have that K is an \mathbf{R} -field, i.e. $K = \mathbf{R}$, $K = \mathbf{C}$ or $K = \mathbf{H}$ the field of quaternions. Note that the group of units $\mathbf{U} = \{u \in K; |u| = 1\}$ is a compact topological group.

All results and proofs as given are valid for modules over a valuated ring $(A, |\cdot|)$ with unit such that A is of characteristic 0 ($\mathbf{Q} \subset A$), $(A, |\cdot|)$ is locally compact and the restriction of the absolute value to \mathbf{Q} is archimedian.

Definition 2. *Let (E, d) be a metric vector space. Given a constant $C_0 \geq 0$ the distance is C_0 -translation invariant if translations are C_0 -isometries, that is for all $x, y, z \in E$,*

$$d(x, y) - C_0 \leq d(x + z, y + z) \leq d(x, y) + C_0.$$

Note that this is equivalent to the right hand side inequality, for all $x, y, z \in E$,

$$d(x + z, y + z) \leq d(x, y) + C_0.$$

Definition 3. *A distance d on E is lipschitz equivalent (or (C_1, C_2) -lipschitz equivalent) to another distance δ on E if there exists $C_1 \geq 1$ and $C_2 \geq 0$ such that for $x, y \in E$,*

$$C_1^{-1}\delta(x, y) - C_2 \leq d(x, y) \leq C_1\delta(x, y) + C_2$$

Definition 4. *A metric vector space (E, d) is metrically normable (MN) if d is lipschitz equivalent to a norm on E .*

Definition 5. Let E be a K vector space. A distance d on E is *lipschitz multiplicative* (or (C_1, C_2, C_3) -lipschitz multiplicative) if there are three constants, $C_1 \geq 1$, $C_2 \geq 0$ and $C_3 \geq 0$, such that for any $\lambda \in K$, $x, y \in E$, we have

$$C_1^{-1} |\lambda| d(x, y) - C_2 |\lambda| - C_3 \leq d(\lambda x, \lambda y) \leq C_1 |\lambda| d(x, y) + C_2 |\lambda| + C_3.$$

Notice that a MN vector space is lipschitz multiplicative. More precisely, if d is (C_1, C_2) -lipschitz equivalent to a norm, then d is $(C_1^2, C_1 C_2, C_2)$ -lipschitz multiplicative.

We denote by μ the (right invariant) Haar measure on the compact group (\mathbf{U}, \cdot) normalized to have total mass 1.

Theorem 2. Let (E, d) be a metric vector space over K with the distance d C_0 -translation invariant and (C_1, C_2, C_3) -lipschitz multiplicative.

Let E_0 be the maximal subspace of E where the distance d is bounded. Then E_0 is a closed subspace of E and the quotient is a metrizable vector space. The Hausdorff distance on classes modulo E_0 induced by d defines a distance D in the quotient.

The vector space $(E/E_0, D)$ is metrically normable by a norm $\|\cdot\|$ which is (C_1^2, C_2') -lipschitz equivalent to d with $C_2' = C_1 C_2 + C_1 C_3 + C_2$, i.e. for $x, y \in E$,

$$C_1^{-2} d(x, y) - C_2' \leq \|x - y\| \leq C_1^2 d(x, y) + C_2'.$$

Moreover, the norm $\|\cdot\|$ can be defined by

$$\|\bar{x}\| = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbf{U}} d(nux, 0) d\mu(u).$$

In particular, if the distance d is unbounded in all non-trivial subspaces, then $E_0 = \{0\}$ and E is metrically normable with a norm lipschitz equivalent to d .

As mentioned before, we can construct in any metrizable topological vector space a translation invariant distance generating the topology thus the problem of metric normability is reduced by theorem 2 to construct such a distance that satisfies an approximate scalar multiplicative property and that is unbounded in non-trivial subspaces.

Using similar ideas we can characterize completely those distances that are asymptotically isometric to a norm.

Definition 6. Let (E, d) be a vector space over K . The distance d is *asymptotically multiplicative* if for any $C_1 > 1$, there exists $C_2 \geq 0$ and $C_3 \geq 0$ such that for $x, y \in E$, we have

$$C_1^{-1} |\lambda| d(x, y) - C_2 |\lambda| - C_3 \leq d(\lambda x, \lambda y) \leq C_1 |\lambda| d(x, y) + C_2 |\lambda| + C_3.$$

Our last theorem gives a necessary and sufficient condition for a distance to be asymptotically isometric to a norm.

Theorem 3. *Let (E, d) be a metric vector space over K . The distance d is asymptotically isometric to a norm $\|\cdot\|$ if and only if d is asymptotically multiplicative, d is unbounded in non-trivial subspaces and for any $C_1 > 1$ there exists $C_0 \geq 0$ such that for any $n \geq 2$ and $x_1, \dots, x_n, y_1, \dots, y_n \in E$, we have*

$$d\left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) \leq C_1 \sum_{i=1}^n d(x_i, y_i) + nC_0.$$

In that case the norm $\|\cdot\|$ can be obtained as described in theorem 2.

It is not difficult to see that the conditions stated are necessary. The last condition is related to the condition of translation invariance in the first theorem in the following way (more precisely see lemma 1 below): All translations are isometries if and only if for any $x_1, x_2, y_1, y_2 \in E$,

$$d(x_1 + x_2, y_1 + y_2) \leq d(x_1, y_1) + d(x_2, y_2).$$

We first prove theorem 2, then theorem 3 follows along the same lines and finally theorem 1 follows from theorem 3.

1) Proof of theorem 2.

For the first part we only need to assume that d is C_0 -translation invariant.

Proposition 1. *Let (E, d) be a metric vector space such that d is C_0 -translation invariant. Then for any $x, y \in E$ the following limit exists*

$$\delta(x, y) = \lim_{n \rightarrow +\infty} \frac{1}{n} d(nx, ny),$$

and we have

$$\delta(x, y) \leq d(x, y) + 2C_0.$$

Lemma 1. *Let (E, d) be a metric vector space. If the distance d is C_0 -translation invariant then we have for any $x_1, x_2, y_1, y_2 \in E$,*

$$d(x_1 + x_2, y_1 + y_2) \leq d(x_1, y_1) + d(x_2, y_2) + 2C_0.$$

Conversely, if we have the previous inequality then the distance d is $2C_0$ -translation invariant.

Proof Lemma 1. We have

$$\begin{aligned} d(x_1 + x_2, y_1 + y_2) &\leq d(x_1, y_1 + y_2 - x_2) + C_0 \\ &\leq d(x_1, y_1) + d(y_1, y_1 + y_2 - x_2) + C_0 \\ &\leq d(x_1, y_1) + d(x_2 + (y_1 - x_2), y_2 + (y_1 - x_2)) + C_0 \\ &\leq d(x_1, y_1) + d(x_2, y_2) + 2C_0 \end{aligned}$$

Conversely, the inequality with $x_2 = y_2$ proves that d is $2C_0$ -translation invariant. \diamond

Proof of proposition 1. Consider for $n \geq 0$,

$$a_n = d(nx, ny) + 2C_0.$$

Lemma 1 shows that the sequence $(a_n)_{n \geq 0}$ is sub-additive: For $n, m \geq 0$,

$$a_{n+m} = d(nx + mx, ny + my) + 2C_0 \leq d(nx, ny) + d(mx, my) + 4C_0 = a_n + a_m.$$

Thus we have (see lemma 3 below for a more general result)

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} a_n = \limsup_{n \rightarrow +\infty} \frac{1}{n} a_n.$$

Also using n times the inequality from lemma 1 we have

$$a_n = d(nx, ny) + 2C_0 \leq nd(x, y) + 2(n + 1)C_0.$$

Thus $(a_n/n)_{n \geq 0}$ is a bounded sequence and has a limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} a_n = \lim_{n \rightarrow +\infty} \frac{1}{n} d(nx, ny) \leq d(x, y) + 2C_0.$$

◇

Remark.

We need only to use the inequality of lemma 1 for "large" x 's and y 's. This will be exploited in the proof of theorem 2.

Lemma 2. *Let (E, d) be a metric vector space over K with the distance d C_0 -translation invariant and (C_1, C_2, C_3) -lipschitz multiplicative. We define*

$$d_0(x, y) = \int_{\mathbf{U}} d(ux, uy) d\mu(u).$$

Then d_0 is a distance $(C_1, C_2 + C_3)$ -lipschitz equivalent to d , C_0 -translation invariant and (C_1, C_2, C_3) -lipschitz multiplicative. Moreover, d and d_0 define the same topology on E .

Proof. Obviously d_0 satisfies the triangle inequality by averaging triangle inequalities. We have for $x, y, z \in E$,

$$\begin{aligned} d_0(x + z, y + z) &\leq \int_{\mathbf{U}} d(u(x + z), u(y + z)) d\mu(u) \\ &\leq \int_{\mathbf{U}} (d(ux, uy) + C_0) d\mu(u) \\ &\leq d_0(x, y) + C_0. \end{aligned}$$

Thus d_0 is C_0 -translation invariant. Also

$$\begin{aligned} d_0(\lambda x, \lambda y) &\leq \int_{\mathbf{U}} d(u\lambda x, u\lambda y) d\mu(u) \\ &\leq C_1 |\lambda| \int_{\mathbf{U}} d(ux, uy) d\mu(u) + C_2 |\lambda| + C_3 \\ &\leq C_1 |\lambda| d_0(x, y) + C_2 |\lambda| + C_3, \end{aligned}$$

and the reverse inequality follows in the same way. Finally the integration over $u \in \mathbf{U}$ of

$$C_1^{-1} d(x, y) - C_2 - C_3 \leq d(ux, uy) \leq C_1 d(x, y) + C_2 + C_3,$$

shows that d_0 is $(C_1, C_2 + C_3)$ -Lipschitz equivalent to d .

The distances d and d_0 define the same topology. Let (x_n) such that $d(x_n, x_0) \rightarrow 0$. Then for all $u \in \mathbf{U}$ we have $d(ux_n, ux_0) \rightarrow 0$ and the sequence of functions $u \mapsto d(ux_n, ux_0)$ are uniformly bounded (the sequence (x_n) is d -bounded and \mathbf{U} is compact). Thus by

Lebesgue dominated convergence we have that $d_0(x_n, x_0) \rightarrow 0$. Conversely let (x_n) such that $d_0(x_n, x_0) \rightarrow 0$. Then there is a sequence $u_n \in \mathbf{U}$ such that $d(u_n x_n, u_n x_0) \rightarrow 0$. Since \mathbf{U} is compact we can extract a sub-sequence such that $u_n \rightarrow u$. Then $d(u x_n, u x_0) \rightarrow 0$ so $d(x_n, x_0) \rightarrow 0$. \diamond

Definition 7. Under the assumptions of theorem 2 we define

$$\delta_0(x, y) = \lim_{n \rightarrow +\infty} \frac{1}{n} d_0(nx, ny)$$

where d_0 is defined in lemma 2, and

$$\delta(x, y) = \lim_{n \rightarrow +\infty} \frac{1}{n} d(nx, ny).$$

Lemma 3. We have that δ and δ_0 are C_0 -translation invariant.

Proof. It is straightforward from the C_0 -translation invariance of d and d_0 . \diamond

Proposition 2. We have that δ and δ_0 satisfy the triangle inequality and are symmetric. Also if $\lambda \in K$ we have,

$$\delta_0(\lambda x, \lambda y) = |\lambda| \delta_0(x, y).$$

Lemma 4. Let \mathbf{Q}_+ be the set of non-negative rational numbers. The subset $\mathbf{Q}_+ \mathbf{U}$ is dense in K .

Proof of Lemma 4. Given $\lambda \in K$, $\lambda \neq 0$, we have $u = \lambda/|\lambda| \in \mathbf{U}$. Since the restriction of $|\cdot|$ to \mathbf{Q} is archimedian, we have that $|\mathbf{Q}| = |\mathbf{Q}_+|$ is dense in \mathbf{R} , thus in $|K|$. So there exists a sequence of positive rationals (p_n/q_n) such that $p_n/q_n \rightarrow |\lambda|$. We conclude that

$$\lim_{n \rightarrow +\infty} \frac{p_n}{q_n} u = \lambda.$$

\diamond

Proof of proposition 2. The triangle inequality and the symmetry is immediate from the definition.

Given an integer $p \geq 0$ we have

$$\delta_0(px, py) = \lim_{n \rightarrow +\infty} \frac{1}{n} d_0(np x, np y) = p \lim_{n \rightarrow +\infty} \frac{1}{np} d_0(np x, np y) = p \delta_0(x, y).$$

Now if $p/q \in \mathbf{Q}$, $q \geq 1$, $p \geq 1$ we have

$$q \delta_0(p/q x, p/q y) = pq \delta_0(1/q x, 1/q y) = p \delta_0(x, y).$$

so for any rational number $r \in \mathbf{Q}_+$,

$$\delta_0(rx, ry) = |r| \delta_0(x, y).$$

◇

Proposition 3. *We have*

$$\delta_0(x, y) = \int_{\mathbf{U}} \delta(ux, uy) d\mu(u).$$

Proof of proposition 3. From Lemma 1 and compactness of \mathbf{U} we have that

$$\frac{1}{n}d(nux, nuy) \leq d(ux, uy) + 2C_0 \leq M(x, y),$$

where $M(x, y)$ is a bound uniform on n . Thus the functions $u \mapsto \frac{1}{n}d(nux, nuy)$ are uniformly bounded. From Lebesgue's dominated convergence theorem we have

$$\begin{aligned} \delta_0(x, y) &= \lim_{n \rightarrow +\infty} \frac{1}{n}d_0(nx, ny) \\ &= \lim_{n \rightarrow +\infty} \int_{\mathbf{U}} d(unx, uny) d\mu(u) \\ &= \lim_{n \rightarrow +\infty} \int_{\mathbf{U}} d(nux, nuy) d\mu(u) \\ &= \int_{\mathbf{U}} \lim_{n \rightarrow +\infty} d(nux, nuy) d\mu(u) \\ &= \int_{\mathbf{U}} \delta(ux, uy) d\mu(u) \end{aligned}$$

q.e.d.◇

Lemma 5. *Any closed (resp. open) set for δ_0 is closed (resp. open) set for d . So the topology generated by d is richer than the topology generated by δ_0 .*

Proof of Lemma 5. Let x_n be a sequence of points in E such that $d(x_n, x) \rightarrow 0$. We have

$$0 \leq \delta_0(x_n, x) \leq C_1 d(x_n, x) + C_2.$$

Therefore

$$\limsup_{n \rightarrow +\infty} \delta_0(x_n, x) \leq C_2.$$

By proposition 2, for all $\lambda \in K$,

$$\delta_0(x_n, x) = \frac{1}{|\lambda|} \delta_0(|\lambda|x_n, |\lambda|x) \leq \frac{1}{|\lambda|} (C_1 \delta_0(|\lambda|x_n, |\lambda|x) + C_2).$$

Thus

$$0 \leq \limsup_{n \rightarrow +\infty} \delta_0(x_n, x) \leq \frac{C_2}{|\lambda|}$$

for all $\lambda \in K$. Taking limit when $|\lambda|$ tends to $+\infty$ we obtain

$$0 \leq \limsup_{n \rightarrow +\infty} \delta_0(x_n, x) \leq 0.$$

Thus $\lim_{n \rightarrow +\infty} \delta_0(x_n, x) = 0$ and the lemma follows. \diamond

Proposition 4. *Let*

- (i) $E_0 = \{x \in E; \text{for all } u \in \mathbf{U}, \delta(ux, 0) = 0\}$,
 - (ii) $E_1 = \{x \in E; \delta_0(x, 0) = 0\}$,
 - (iii) E_2 maximal subspace where d_0 is bounded,
 - (iv) E_3 maximal subspace where d is bounded,
- Then $E_0 = E_1 = E_2 = E_3$.

Proof of proposition 4. We have

$$\delta_0(x, 0) = \int_{\mathbf{U}} \delta(ux, 0) d\mu(u),$$

thus $E_0 = E_1$. Observe that E_1 is a subspace of E , for $x, y \in E_1$, using the translation invariance,

$$\delta_0(x + y, 0) \leq \delta_0(x, 0) + \delta_0(y, 0) = 0 + 0 = 0.$$

Also $\delta_0(\lambda x, 0) = \lambda \delta_0(x, 0) = 0$. Moreover for $x, y \in E_1$ we have

$$d_0(x, y) \leq C_1 (\delta_0(x, y) + C_2) \leq C_1 (\delta_0(x, 0) + \delta_0(y, 0) + C_2) \leq C_1 C_2.$$

Thus d_0 is bounded in E_1 and $E_1 \subset E_2$. From the definition of δ_0 it follows that $E_2 \subset E_1$, thus $E_1 = E_2$.

Finally $E_2 = E_3$ because d_0 and d are lipschitz equivalent. \diamond

Proposition 5. *The subspace E_0 is a closed subspace of E .*

Proof of proposition 5. let (x_n) be a converging sequence of points in E_0 , $x_n \rightarrow x$. If $x \notin E_0$ then $\delta_0(x, 0) \neq 0$. Let $C = C_1 C_2 + C_1 C_3 + C_2$. Consider

$$y_n = \frac{C + 1}{\delta_0(x, 0)} x_n.$$

We have $y_n \in E_0$ (since E_0 is a subspace), $y_n \rightarrow y$ with

$$\delta_0(y, 0) = \frac{C + 1}{\delta_0(x, 0)} \delta_0(x, 0) > C.$$

Then

$$\begin{aligned} C < \delta_0(y, 0) &\leq \delta_0(y - y_n, 0) + \delta_0(y_n, 0) = \delta_0(y - y_n, 0) \\ &\leq C_1 d_0(y, y_n) + C_2 \leq C_1^2 d(y, y_n) + C_1(C_2 + C_3) + C_2 = C_1^2 d(y, y_n) + C. \end{aligned}$$

Passing to the limit $n \rightarrow +\infty$, we get $C < C$. Contradiction. \diamond

Proposition 6. *We define for a class $\bar{x} \in E/E_0$,*

$$\|\bar{x}\| = \delta_0(x, 0).$$

The definition is independent of the representant x of the class $\bar{x} = x + E_0$ and $\|\cdot\| : E/E_0 \rightarrow \mathbf{R}_+$ is a norm.

Proof of proposition 6. If $y \in \bar{x}$ then $x - y \in E_0$ thus $\delta_0(x - y, 0) = 0$ so by translation invariance $\delta_0(x, y) = 0$ and

$$\delta_0(x, 0) \leq \delta_0(x, y) + \delta_0(y, 0) = \delta_0(y, 0).$$

In the same way $\delta_0(y, 0) \leq \delta_0(x, 0)$ and finally $\delta_0(x, 0) = \delta_0(y, 0)$. Also if $\|\bar{x}\| = 0$ then $\delta_0(x, 0) = 0$ and $x \in E_0$, i.e. $\bar{x} = \bar{0}$. The other properties of a norm follow from the properties of δ_0 . \diamond

Definition 8. *We denote by $D : E/E_0 \rightarrow \mathbf{R}_+$, resp. $D_0 : E/E_0 \rightarrow \mathbf{R}_+$, the Hausdorff distances for d , resp. d_0 , between classes modulo E_0 ,*

$$D(x_0 + E_0, y_0 + E_0) = \max \left(\inf_{x \in \bar{x}_0} \sup_{y \in \bar{y}_0} d(x, y), \inf_{y \in \bar{y}_0} \sup_{x \in \bar{x}_0} d(x, y) \right).$$

Hausdorff distances over non-compact sets do not need to be finite. Also when the distance is zero the sets do not need in general to coincide. In our situation Hausdorff distances define a proper distance on classes modulo E_0 .

Lemma 6. *The spaces $(E/E_0, D)$ and $(E/E_0, D_0)$ are metric spaces, and D and D_0 define the quotient topology on E/E_0 .*

Proof of lemma 6. We first prove that D and D_0 do define distances. We carry the proof for D . The same proof applies to D_0 . If $x \in \bar{x}_0$ and $y \in \bar{y}_0$ we have

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq d(x_0, y_0) + d(x - x_0, 0) + d(y_0 - y, 0) + 2C_0,$$

and the last two terms are uniformly bounded since $x - x_0 \in E_0$ and $y_0 - y \in E_0$. This shows that $D(\bar{x}_0, \bar{y}_0) < +\infty$.

Assume that $D(\bar{x}_0, \bar{y}_0) = 0$. Then there is a sequence (e_n) with $e_n \in E_0$ such that $y_0 + e_n \rightarrow x_0$. Therefore $e_n \rightarrow x_0 - y_0$. But we have proved that E_0 is closed, thus $x_0 - y_0 \in E_0$ and $\bar{x}_0 = \bar{y}_0$.

We denote $\pi : E \rightarrow E/E_0$ the quotient map. We prove that each open set U for the quotient topology of E/E_0 is open for D . Let $\bar{x}_0 \in U$ and $D(\bar{x}_n, \bar{x}_0) \rightarrow 0$. We have to prove that there exists N such that for $n \geq N$, $\bar{x}_n \in U$. We have that $x_n + E_0 \rightarrow x_0 + E_0$ in Hausdorff metric for d . Thus there exists a sequence $x'_n \in x_n + E_0$ such that $d(x'_n, x_0) \rightarrow 0$. Since d defines the topology of E and $\pi^{-1}(U)$ is open there exists N such that for $n \geq N$ we have $x'_n \in \pi^{-1}(U)$. Then $\bar{x}_n = \bar{x}'_n = \pi(x'_n) \in U$. q.e.d. \diamond

Proposition 7. *The norm $\|\cdot\|$ is $(C_1, C_2 + C_3)$ -lipschitz equivalent to D_0 and (C_1^2, C_2') -lipschitz equivalent to D .*

Proof of proposition 7. We have for $x \in \bar{x}_0, y \in \bar{y}_0$,

$$C_1^{-1}d_0(x, y) - (C_2 + C_3) \leq \delta_0(x, y) = \delta_0(x_0, y_0) = \|\bar{x}_0 - \bar{y}_0\| \leq C_1d_0(x, y) + (C_2 + C_3).$$

Now letting x and y run over \bar{x}_0 and \bar{y}_0 respectively, and using the definition of D we have the result. Same proof for D_0 . \diamond

2) Proof of theorem 3.

The conditions are necessary.

We assume that d is asymptotically isometric to a norm $\|\cdot\|$. Let $C_1 > 1$. Then $C_1^{1/2} > 1$ and there exists $C_0 \geq 0$ such that for $x, y \in E$,

$$C_1^{-1/2}\|x - y\| - C_0 \leq d(x, y) \leq C_1^{1/2}\|x - y\| + C_0.$$

Then for any $\lambda \in K$ we have,

$$\begin{aligned} d(\lambda x, \lambda y) &\leq C_1^{1/2}\|\lambda x - \lambda y\| + C_0 = C_1^{1/2}|\lambda| \|x - y\| + C_0 \\ &\leq C_1^{1/2}|\lambda| \left(C_1^{1/2}d(x, y) + C_1^{1/2}C_0 \right) + C_0 \\ &\leq C_1|\lambda| d(x, y) + C_1C_0|\lambda| + C_0 \end{aligned}$$

The reverse inequality is proved in the same way and d is asymptotically multiplicative. Also for $x_1, \dots, x_n, y_1, \dots, y_n \in E$ we have

$$\begin{aligned} d\left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) &\leq C_1^{1/2}\left\|\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right\| + C_0 \\ &\leq C_1^{1/2}\sum_{i=1}^n \|x_i - y_i\| + C_0 \\ &\leq C_1\sum_{i=1}^n d(x_i, y_i) + nC_1C_0 + C_0 \\ &\leq C_1\sum_{i=1}^n d(x_i, y_i) + n(2C_1C_0) \end{aligned}$$

thus the condition in theorem 2 is necessary.

The conditions are sufficient.

We construct the norm by the same strategy as in theorem 1. We need a refinement on the lemma on sub-additive sequences.

Lemma 7. *Let $(a_n)_{n \geq 0}$ be a sequence of real numbers satisfying the following weak sub-additive property: For any $C_1 > 1$ there exists $C_0 \geq 0$ such that for any $q \geq 2$ and any $m_1, \dots, m_q \geq 0$,*

$$a_{m_1 + \dots + m_q} \leq C_1 \sum_{i=1}^q a_{m_i} + qC_0.$$

Then

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} a_n = \limsup_{n \rightarrow +\infty} \frac{1}{n} a_n.$$

Proof. Fix for the moment $C_1 > 1$. Choose $n \geq 1$. For any $m \geq 0$ we can consider the euclidian division $m = nq + r$, with $0 \leq r < n$. We have

$$a_m = a_{nq+r} \leq C_1(qa_n + a_r) + (q+1)C_0.$$

Dividing by m and taking the least upper bound for $m \rightarrow +\infty$ ($q \rightarrow +\infty$) we have

$$\limsup_{m \rightarrow +\infty} \frac{1}{m} a_m \leq C_1 \frac{1}{n} a_n + \frac{C_0}{n}.$$

Now taking the greater lower bound for $n \rightarrow +\infty$ we get

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} a_n \leq C_1 \liminf_{n \rightarrow +\infty} \frac{1}{n} a_n.$$

Since this holds for any $C_1 > 1$ the lemma follows. \diamond

It is simple to check that we have the same lemma as lemma 2 for theorem 1:

Lemma 8. *We assume the hypothesis of theorem 2. We define*

$$d_0(x, y) = \int_{\mathbf{U}} d(ux, uy) \, d\mu(u) .$$

Then d_0 is a distance asymptotically equivalent to d and satisfying the hypothesis of theorem 3.

Now we have:

Proposition 8. *For any $x, y \in E$, the limit*

$$\delta(x, y) = \lim_{n \rightarrow +\infty} \frac{1}{n} d_0(nx, ny)$$

exists.

Proof. The sequence

$$a_n = d_0(nx, ny)$$

is weakly sub-additive:

$$\begin{aligned} a_{m_1+\dots+m_q} &= d_0((m_1 + \dots + m_q)x, (m_1 + \dots + m_q)y) \\ &= d_0(m_1x + \dots + m_qx, m_1y + \dots + m_qy) \\ &\leq C_1 \sum_{i=1}^q a_{m_i} + qC_0. \end{aligned}$$

Moreover a_n/n is bounded:

$$\begin{aligned} a_n &= d(nx, ny) = d(x + \dots + x, y + \dots + y) \\ &\leq C_1 \sum_{i=1}^n d(x, y) + nC_0 \\ &= C_1 nd(x, y) + nC_0. \end{aligned}$$

Thus

$$\frac{a_n}{n} \leq C_1 d(x, y) + C_0.$$

The result follows from lemma 7. \diamond

Proposition 9. *For $x, y \in E$ we define*

$$\delta_0(x, y) = \lim_{n \rightarrow +\infty} \frac{1}{n} d_0(nx, ny).$$

Then δ_0 is translation invariant. If we define

$$||x - y|| = \delta_0(x, y)$$

then $||\cdot||$ is a norm that is asymptotically isometric to d_0 , so also to d .

Proof. We prove the translation invariance. The rest follows the same lines as the proof of proposition 2. Let $x, y, z \in E$. For any $C_1 > 1$, there exists $C_0 \geq 0$ such that

$$\begin{aligned} d_0(n(x+z), n(y+z)) &= d_0(nx+nz, ny+nz) \\ &\leq C_1 d_0(nx, ny) + C_1 d_0(nz, nz) + 2C_0 \\ &= C_1 d_0(nx, ny) + 2C_0. \end{aligned}$$

Now dividing by n and passing to the limit $n \rightarrow +\infty$, we get, for any $C_1 > 1$,

$$\delta_0(x+z, y+z) \leq C_1 \delta_0(x, y).$$

Therefore making $C_1 \rightarrow 1$, for all $x, y, z \in E$,

$$\delta_0(x+z, y+z) \leq \delta_0(x, y).$$

Replacing x by $x+z$, y by $y+z$ and z by $-z$, we get the opposite inequality, and the translation invariance. \diamond

This finishes the proof of theorem 3.

Theorem 3 implies Theorem 1.

Let (E, d) be asymptotically isometric to an AMN vector space. Therefore d is unbounded in non-trivial subspaces and we have seen that it satisfies the other hypothesis of theorem 3. Thus d is asymptotically isometric to a norm of E .

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